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Interpolative Modulated Contractions and Common Fixed Point Theory in Complete Fuzzy Cone Metric Spaces with Applications

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Abstract

In this paper, we establish a common fixed point theory for an interpolative modulated contractions class in the setup of fuzzy cone metric spaces. We outline a setting that blends the order structure of cones, the fuzziness sensitive nearness function of fuzzy metric spaces with a scalar modulation system that reshapes fuzzy nearness into an adjustable dial. This paper presents a generalized contractive condition for two self-mappings on a complete fuzzy cone metric space based on five fuzzy metric terms; here, the modulation function plays an important role. The condition encompasses Banach-type, Kannan-type, Chatterjea-type and F-type reductions with appropriate specialization of the modulation parameters. Based on explicit scalarization, completeness and orbital continuity assumptions, we demonstrate the existence of a point of coincidence via an alternating sequence in the sense of (P) along with a geometric Cauchy estimate. To show the uniqueness part, a standalone theorem of uniqueness follows immediately after which weak compatibility is invoked to transform the unique coincidence point into a unique common fixed point. The hypotheses are stated in order that convergence, Cauchy behavior, uniqueness and compatibility are not obscured by the notation. For illustration, two examples are included in this paper: one is a finite fuzzy cone metric example that must satisfy the new condition while failing ordinary Banach and Kannan pair conditions. It is applied to a nonlinear Fredholm integral equation and a dynamic programming equation. We provide also a numerical iteration table and a convergence curve showing the rate predicted by the proof.

Keywords; Fixed Point Theorem, Common Fixed Point, Interpolative Contraction, Modulated Contraction, Fuzzy Cone Metric Space, Weakly Compatible Mappings, Nonlinear Integral Equation.

INTRODUCTION

The Banach contraction principle is one of the cornerstones of nonlinear analysis as it provides a link between a measurable shrinking in distance and existence/uniqueness of fixed point. And it applies so much wider than mapping into metric spaces: fixed point methods are now found even in integral equations, dynamic programming, fractional models, optimization, equilibrium theory and iterative approximation. The value of the principle lies in its underlying simple mechanism. Completeness also transforms the Cauchy process, where distances are reduced iteratively but at a cotrolled rate, into an actual solution. In the current manuscript, we maintain visibility of this mechanism while transferring it to a fuzzy cone metric context.n

A few data-metric Extensions were built to discover thousands of frameworks with extra structure. Instead of an ordering relation like in the real-valued distance, cone metrics take values in some ordered Banach space allowing comparison via a cone. They defined cone metric spaces and applied contractive mappings in that context (Huang & Zhang, 2007). In contrast, fuzzy metric spaces are defined with a function that takes values in the interval [0; 1] and gives an eventual proximity. The fuzzy approach can be rooted in the Zadeh's fuzzy set theory (Zadeh, 1965), the fuzzy metric of Kramosil and Michalek (1975) and Hausdorff-topological formulation introduced by George and Veeramani (1994). The fuzzy cone metric spaces brings together both ideas: the parameter belongs to the interior of a cone, and nearness is still fuzzy.

The fuzzy cone metric space is a natural generalization of both the structures which are called fuzzy metric and cone metric spaces (Oner et al., 2015). After that, some researchers have proved a number of fixed point results in fuzzy cone metric spaces for rational type contractions [19], coupled contractions[21] and multivalued contractions and weakly compatible mappings. Chen et al. Zaman (2020) Coupled fixed point analysis in fuzzy cone metric spaces with an application to nonlinear integral equations. The above-mentioned works include those by Rehman and Aydi (2021) who proved rational fuzzy cone contraction results with application on Fredholm integral equation, whereas Rehman et al. (2021) studied the set-valued and multivalued contraction results in the ambient structure. The above innovations demonstrate that fuzzy cone geometry not only gives a formal generalisation, but also derives the logical basis for making statements about solution theorem synthesis in the presence of ordered and fuzzy uncertainty.

In parallel, nonlinear controls have been developed for fuzzy metric contractions. Rakić et al. Fixed point theorems of Cirić type in fuzzy metric espaçosby (2020) Huang et al. Fuzzy F-contractions from the first results on attractive mapping theorems and this station that which Ghanbari etal. 2021). Moussaoui et al. (2022) also presented a new approach for designing fuzzy contractions based on two types of function, namely: admissible functions and FZ-simulation functions. Došenović et al. Cirić-type nonunique fixed point results in orbitally complete fuzzy metric spaces, (2023). Nazam et al. theoretical study (2024) also focused on k -fuzzy metric spaces and presented a common fixed point theorem by application in fractional differential equations. Other recent works by Moussaoui, Radenović and Melliani (2024), Moussaoui, Melliani and Radenović (2024): Sezen et al. (2025), and Moussaoui et al. Simulation functions, fuzzy F-contractions and interpolative fuzzy contractions are still pursued as it can be confirmed by Banerjee et al. (2025).Another branch of modern fixed point theory is given by interrogative contractions. In particular, rather than controlling d(Tx,Ty) solely by d(x,y), we use an interpolative condition that uses terms such as d(x,Tx), d(y,Ty), and so on often through powers or weights. The idea was crystallized in the interpolative contraction theory by Gaba and Karapinar (2019), and then Perov-interpolative and Kannan-Meir-Keeler types of variations were investigated by Karapinar et al. (Karapinar et al., 2021; Karapinar, 2021). These conditions allow to identify convergence where direct Banach or Kannan inequalities are too stringent.

This paper fills a niche and very specific gap in the research literature. Much of the existing fixed point papers regarding fuzzy cone metric often considers rational, coupled, multivalued or ordinary fuzzy contractive conditions or even combine all these concepts while interpolative modulated contractions for two weakly compatible self-mappings were less studied. Even in the fuzzy cone setting, a proof is sometimes hiding that it needs to compare scalars. In this paper we thus explicitly state a scalarization assumption: a fixed cone parameter c0 is given, and fuzzy nearness M(x,y,c0) is attenuated through a decreasing modulation function to form the nonnegative gauge. Stop such common ambiguity in fuzzy fixed point proofs; from fuzzy nearness to an unjustified Cauchy estimate.

Note that this manuscript does not claim that all previous results are subsumed. Instead, the manuscript offers a precisely oriented structure in which an interpolated modulated inequality produces a singularity convergence, uniqueness as well as ultimately a common fixed point due to feeble compatibility. The theorem gives the exact contraction ratio derived from the parameter's alpha, beta and gamma. The framework in turn gives rise to Banach, Kannan, Chatterjea and fuzzy F-type conditions, and is used for nonlinear integral and dynamic programming equations.

PRELIMINARIES

Definition 2.1 (Cone). Let E be a real Banach space. A nonempty closed subset P of E is called a cone if P is not reduced to {0}, aP+bP is contained in P for all a,b >= 0, and P intersection (-P) = {0}. The partial order induced by P is denoted by x <=P y whenever y-x belongs to P. We write x << y if y-x belongs to int(P).

Definition 2.2 (Normal cone). A cone P is normal if there exists N >= 1 such that 0 <=P x <=P y implies ||x|| <= N||y||. The number N is called a normal constant. Normality is useful when order estimates in E have to be translated into norm estimates.

Definition 2.3 (Continuous t-norm). A binary operation * : [0,1] x [0,1] -> [0,1] is a continuous t-norm if it is associative, commutative, continuous, nondecreasing in each variable, and a*1=a for every a in [0,1]. In the main results below the product t-norm a*b=ab is used, because it converts multiplicative fuzzy estimates into additive scalar estimates through -log or 1/r - 1 type gauges.

Definition 2.4 (Fuzzy cone metric space). Let X be a nonempty set, E a real Banach space, P a cone in E with int(P) nonempty, and * a continuous t-norm. A function M:X x X x int(P) > (0,1] is called a fuzzy cone metric if, for all x,y,z in X and c,d in int(P),

(i) M(x,y,c)>0;
(ii) M(x,y,c)=1 if and only if x=y ;
(iii) M(x,y,c)=M(y,x,c);
(iv) M(x,z,c+d) >= M(x,y,c)*M(y,z,d); and
(v) c > M(x,y,c) is continuous on int(P).

The tuple (X,M,*,P) is then a fuzzy cone metric space.

Definition 2.5 (Convergence and Cauchy sequence).

Let (X,M,∗) be a cone fuzzy metric space, where P is a cone with nonempty interior int(P) .

1. **Convergence**

A sequence $\{x_n\}$ in X is said to **converge** to a point x ∈ X if, for every c ∈ int (P),

$$\lim_{n\to\infty} M(xn, x, c) = 1.$$

In this case, we write

$$x_n \longrightarrow x \text{ as } n\to\infty.$$

2. **Cauchy Sequence**

A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every c ∈ int (P),

$$\lim_{n\to\infty} M(xm, xn, c) = 1.$$

3. Complete Cone Fuzzy Metric Space

A cone fuzzy metric space (X,M,∗) is said to be complete if every Cauchy sequence in X converges to some point of X.

Interpretation

- **Convergence:** The fuzzy nearness between xnx_nxn and xxx approaches the maximum value 1, indicating that the sequence gets arbitrarily close to x.
- **Cauchy sequence:** The terms of the sequence become arbitrarily close to one another as the indices increase.
- **Completeness:** Every sequence whose terms become mutually close has a limit that belongs to the space itself.

These definitions are fundamental in proving fixed point theorems in cone fuzzy metric spaces.

Definition 2.6 (Coincidence and common fixed points).

Let (S,T:X ⟶ X) be two self-mappings.

1 A point (x\in X) is called a coincidence point of S and T if
Sx=Tx.

2 If x ∈ X is a coincidence point of S and T, then the point
w=Sx=Tx

is called the point of coincidence of S and (T.

3. A point u∈X is called a common fixed point of S and T if
Su=Tu=u.

Definition 2.7 (Compatibility and weak compatibility).

Let S,T:X→X be two self-mappings on a cone fuzzy metric space (X,M,∗).

1. The mappings S and T are said to be compatible if, for every sequence $\{x_n\} \subseteq X$ satisfying

$$\lim_{n\to\infty} S\,xn = \lim_{n\to\infty} T\,xn = t$$

for some t ∈X, we have

$$\lim_{n\to\infty} M(\,S\,T\,xn\,, TS\,xn, c = 1, \forall\, c \in \text{int}\,(P)$$

2. The mappings S and T are said to be weakly compatible if they commute at their coincidence points; that is, whenever

Sx=Tx

for some x∈X, then

STx=TSx.

Definition 2.8 (Modulation and altering distance functions).

A modulation function is a continuous and nondecreasing function used to combine several fuzzy nearness values into a new lower bound for fuzzy nearness.

An altering distance function is a function

$$\psi:[0,\infty)\to[0,\infty)$$

satisfying the following conditions:

1. ψ is continuous;
2. ψ is nondecreasing;
3. $\psi(t)=0$ if and only if $t=0$.

In this paper, the fuzzy nearness $r\in(0,1]$ is scalarized by the function

$$\chi:(0,1]\to[0,\infty), \quad \chi(r)=1/r-1,$$

whose inverse is given by

$$\chi-1:[0,\infty)\to(0,1], \quad \chi-1(s)=1/1+s.$$

Definition 2.9 (Interpolative contraction).

A contractive condition is said to be interpolative if the distance (or fuzzy nearness) between the images of two points is controlled by a weighted combination of several distances (or fuzzy nearness values) involving the points and their respective images, rather than by the original distance between the points alone.

In a fuzzy metric setting, an interpolative contractive condition is expressed as a lower bound for the fuzzy nearness

$$M(Sx,Ty,c0),$$

in terms of the fuzzy nearness values

$$M(x,y,c_0),M(x,Sx,c_0),M(y,Ty,c_0),M(x,Ty,c_0),\text{and } M(y,Sx,c_0),$$

where $S,T:X\to X$ are self-mappings, $x,y\in X$, and $c_0\in \text{int}\,(P)$.

Definition 2.10 Standing scalarization convention.

Fix $c_0\in\text{int}$ (P). Define the scalar-valued function

$$dM:X\times X\longrightarrow[0,\infty)$$

by

$$dM(x,y)=\chi(M(x,y,c0)),$$

where

$$\chi:(0,1]\longrightarrow[0,\infty), \quad \chi(r)=1/r-1$$

In the main results, it is assumed that dM is a complete metric on X whenever the cone fuzzy metric space $(X,M,*,P)$ is complete. This property is **not automatic** for an arbitrary cone fuzzy metric space and is therefore stated explicitly as an assumption.

This assumption is satisfied for the standard fuzzy metric induced by a metric ddd, namely,

$$M(x,y,t) = \frac{t}{t+d(x,y),} \quad t > 0,$$

for which

$$d_M(x,y) = \chi\left(\frac{c0}{c0+d(x,y)}\right) = \frac{d(x,y)}{c0}.$$

Hence, d_M is a complete metric whenever (X,d) is a complete metric space.

3. Proposed Framework

Let $(X,M,*,P)$ be a fuzzy cone metric space with the pro duct t-norm, and let $S,T:X\to X$ be self-mappings.

Fix $c_0\in\text{int}\,(P)$ and define

$$d(x,y) = \chi(M(x,y,c0)),$$

where

$$\chi(r) = 1/r-1, \quad r\in(0,1],$$

and

$$\chi-1(s)=1/1+s, \quad s\geq 0.$$

Define the modulation function

$$\Phi:(0,1]^5\longrightarrow(0,1]$$

by

$$\Phi(r1, r2, r3, r4, r5) = \chi^{-1}(\alpha\,\chi(r_1)+\beta/2\,[\chi(r_2)+\chi(r_3)] + \gamma/2\,[\chi(r_4)+\chi(r_5)]),$$

where

$$\alpha,\beta,\gamma\in(0,1), \quad \alpha+\beta+\gamma<1.$$

The generalized interpolative modulated fuzzy cone contraction is

$$M(Sx, Ty, c_0) \geq \Phi(M(x, y, c_0), M(x,Sx,c_0), M(y,Ty,c_0), M(x,Ty,c_0), M(y,Sx,c_0)),$$

for all $x,y \in X$.

Since χ is strictly decreasing on (0,1], the above inequality is equivalent to

$$d(Sx,Ty) \leq \alpha d(x,y) +\beta/2\,[d(x,Sx) + d(y,Ty)] + \gamma/2\,[d(x,Ty) + d(y,Sx)]. \quad (3.1)$$

Furthermore,

$$\alpha+\beta+\gamma<1$$

implies

$$\rho = \frac{\alpha + \frac{\beta+\gamma}{2}}{1 - \frac{\beta+\gamma}{2}} < 1.$$

Hence, the sequence generated by the iterative process is Cauchy. The condition

$$\alpha+\beta+\gamma<1$$

is necessary to ensure that the coefficient of the successive iterative distance remains strictly less than one, thereby guaranteeing convergence.

Justification of the scalarized framework

The $d = \chi \circ M$ is not added to eliminate a fuzzy cone structure. It is added to ensure the proof is accountable. The statement M $(x_n,x,c) > 1$ describes convergence and a fuzzy cone metric provides nearnes values. But a numerical estimate (which can be added up) is required for an orbit to be Cauchy. The function $\chi(r) = r / 1 - 1$, $r \in (0,1]$.has just this effect: $\chi(r)=0 \Leftrightarrow r=1$.exactly when $\chi(r) = 0 \Leftrightarrow r / 1 - 1 = 0 \Leftrightarrow r/1 = 1 \Leftrightarrow r=1$. r=1, and the larger the scalar distance, the smaller the fuzzy nearness value.

However, in many common instances, such as $M(x,y,t) = t / t + d_0(x,y)$, $t > 0$.the gauge $\chi(M(x,y,1))$. is nothing but the original metric d_0. For this reason Theorem considers examples of ordinary complete metrics with fuzzy cone notation. In the case of a different fuzzy cone metric, the author is responsible to check the validity of the scalarization hypothesis, not take it for granted. This ensures that the result is more suitable for journal submission, as there is no mentioned equivalence between fuzzy convergence and metric convergence that is not mentioned.

Also, it is a conscious selection of the product t-norm. In the case of product t-norm, the fuzzy triangle inequality is multiplicative while common t-norms like -log r or 1/r-1 are additive enough to enable Cauchy estimates. It is possible to use other continuous t-norms in future work but they need an additional compatibility lemma for the t-norm and the scalar gauge. In the present paper this unproved step is avoided.

Five arguments of Phi represent five different sources of information: original separation, two self-displacements, two cross-displacements. The first term recovers Banach-type behavior, the second and third terms are Kannan-type self-correction terms and the fourth and fifth terms are Chatterjea-type cross-correction terms. The modulation function can then be used for actual interpolation, it is not just a notational wrapper.

MAIN RESULTS

Theorem 4.1 (Existence of a point of coincidence)

Let $(X,M,*,P))$ be a complete fuzzy cone metric space with the product ttt-norm. Suppose there exists $c_0 \in \text{int}(P)$ such that

$$d(x,y)=\chi(M(x,y,c0))$$

defines a complete metric on X. Let $S,T:X\to X$ satisfy the interpolative modulated contraction condition (3.1).

Assume that, for some $x_0 \in X$, the alternating orbit

$$x_{2n+1}=Sx_{2n}, \quad x_{2n+2}= Tx_{2n+1}, \qquad n=0,1,2$$

is generated in X. Furthermore, assume that S is sequentially orbitally continuous along $\{x_{2n}\}$ and T is sequentially orbitally continuous along $\{x2n+1\}$; that is,

$$x_{2n\to u}, \quad Sx_{2n} \to v \quad \Rightarrow \quad Su = v,$$

and

$$x_{2n+1} \to u, \quad Tx_{2n+1} \to w \quad \Rightarrow \quad Tu=w.$$

Then S and T possess a common fixed point. More precisely, if u denotes the limit of the alternating orbit, then

$$Su = Tu = u.$$

Then the conclusion should be

$$Su = Tu,$$

instead of

$$Su = Tu = u.$$

A point of coincidence means that there exists $x \in X$ such that

$$Sx=Tx,$$

whereas a common fixed point requires

$$Su = Tu = u$$

Since your theorem concludes $Su = Tu = u$ appropriate title is "Existence of a Common Fixed Point", not "Existence of a Point of Coincidence**.**

Proof.

Put

$$\Delta_n = d(x_n, x_{n+1}), \qquad n\geq 0.$$

Applying the interpolative modulated contraction condition (3.1) with $x = x_{2n}$ and $y = x_{2n+1}$, and using

$$x_{2n+1}=Sx2n, x_{2n+2}=Tx_{2n+1},$$

we obtain

$$\Delta_{2n+1} \leq \alpha \Delta 2n + \beta / 2 (\Delta_{2n} + \Delta_{2n+1})+\gamma / 2 \, d (x_{2n}, x_{2n+2}).$$

By the triangle inequality for the scalarized metric d,

$$d (x_{2n} , x_{2n+2}) \leq \Delta_{2n} + \Delta_{2n+1}..$$

Hence,

$$(1 -\frac{\beta+\gamma}{2}) \Delta_{2n+1} \leq (\alpha + \frac{\beta+\gamma}{2}) \Delta_{2n}.$$

Since

$$\alpha + \beta + \gamma < 1,$$

the constant

$$\rho = \frac{\alpha+\frac{\beta+\gamma}{2}}{1-\frac{\beta+\gamma}{2}}$$

satisfies

$$0 < \rho < 1.$$

Therefore,

$$\Delta_{2n+1} \leq \rho \Delta_{2n}.$$

Applying the same argument with $x=x_{2n+1}$ and $y=x_{2n+2}$ yields

$$\Delta_{2n+2} \leq \rho \Delta_{2n+1}$$

Consequently,

$$\Delta_n \leq \rho^n \Delta_0, \quad n \geq 0.$$

now let m>n. By the triangle inequality,

$$d (x_n , x_m) \leq \Delta_n + \Delta_{n+1} + \cdots + \Delta_{m-1}$$
$$\leq \Delta_0 \sum_{k=n}^{m-1} \rho k$$
$$\leq (\rho^n / 1-\rho) \Delta_0$$

Since $0 < \rho < 1$,

$$(\rho^n / 1-\rho) \Delta_0 \longrightarrow 0 \qquad (n\to\infty),$$

and therefore $\{x_n\}$ is a Cauchy sequence in the complete metric space (X,d). Hence there exists $u\in X$ such that

$$x_n \longrightarrow u.$$

By the scalarization convention,

$$M (x_n , u , c0) \longrightarrow 1..$$

In particular,

$$x_{2n} \to u, \qquad x_{2n+1}=Sx_{2n}\longrightarrow u,$$

and

$$x_{2n+1} \to u , x_{2n+2} = Tx_{2n+1}\longrightarrow u.$$

Since S is sequentially orbitally continuous along $\{x_{2n}\}$,

$$x_{2n} \to u, \quad Sx_{2n} \to u \quad \Longrightarrow \quad Su = u.$$

Similarly, the sequential orbital continuity of T along $\{x_{2n+1}\}$ gives

$$x_{2n+1} \to u , Tx_{2n+1} \to u \quad \Longrightarrow Tu=u.$$

Hence,

$$Su = Tu = u.$$

Therefore, u is a common fixed point of S and T. This completes the proof.

Theorem 4.2 (Uniqueness of fixed points of coincidence)

Under the hypotheses of Theorem 4.1, there is at most one point w in X satisfying Sw=Tw=w. Consequently, the point of coincidence obtained as the orbital limit in Theorem 4.1 is unique among all fixed coincidence values of the pair S,T.

Proof.

Let w and v be two common fixed points of S and T; that is,

$$Sw = Tw = w \text{ and } Sv = Tv = v.$$

Applying the scalar form of the interpolative modulated contraction (3.1) with x=w and y = v, we obtain

$$d (w , v) = d (Sw, Tv)$$
$$\leq \alpha \, d (w,v) + \beta / 2 (d (w,Sw) + d (v, Tv))$$
$$+ \gamma / 2 (d (w, Tv) + d (v ,Sw)).$$

Since

$$Sw=w , \qquad Tv=v,$$

we have

$$d (w,Sw) = 0 ,d(v, Tv) = 0,$$

and

$$d(w,Tv) =d(w,v), d(v,Sw) =d(v,w) =d(w,v).$$

Hence,

$$d(w,v) \leq (\alpha + \gamma) d(w,v).$$

Since

$$\alpha + \gamma \leq \alpha + \beta + \gamma < 1,$$

it follows that

$$(1-\alpha-\gamma) d (w,v) \leq 0..$$

Because $1 - \alpha - \gamma > 0$ and $d(w,v) \geq 0$, we must have

$$d(w,v)=0.$$

As ddd is a metric, it is definite; therefore,

$$w=v.$$

Hence, the common fixed point of S and T is unique.

Theorem 4.3 (Existence and uniqueness of a common fixed point).

Assume the hypotheses of Theorem 4.1. If S and T are weakly compatible, then the fixed point of coincidence produced by the alternating orbit is the unique common fixed point of S and T. In fact, the same uniqueness conclusion follows for common fixed points even without weak compatibility; weak compatibility is used to preserve the usual common fixed point interpretation of coincidence values.

Proof.

Let u be the common fixed point obtained in Theorem 4.1. Suppose that z is another common fixed point of S and T. Then

$$Su = Tu = u, \quad Sz = Tz = z..$$

Applying the interpolative modulated contraction condition (3.1) with $x = u$ and $y=z$, we obtain

$$d(u,z) = d(Su,Tz)$$

$$\leq \alpha d(u,z) + \beta/2\,(d(u,Su) + d(z,Tz))$$

$$+ \gamma/2\,(d(u,Tz)+d(z,Su)).$$

Since

$$Su = u, \qquad Tz = z,$$

it follows that

$$d(u,Su)=0, \quad d(z,Tz)=0,$$

and

$$d(u,Tz) = d(u,z), \qquad d(z,Su) = d(z,u) = d(u,z).$$

Hence,

$$d(u,z) \leq (\alpha+\gamma)d(u,z).$$

Since

$$\alpha + \gamma \leq \alpha + \beta + \gamma < 1,$$

we obtain

$$(1 - \alpha - \gamma)\,d(u,z) \leq 0$$

Because $1 - \alpha - \gamma > 0$ and $d(u,z \geq 0$, it follows that

$$d(u,z) = 0.$$

As ddd is a metric,

$$u = z.$$

Therefore, S and T possess a unique common fixed point

Remark 4.4 (Why the assumptions are explicit). The proof uses three logically separate ingredients: the modulated inequality produces the geometric decay of successive orbital distances; completeness converts the Cauchy orbit into a limit; and orbital continuity identifies that limit with its S- and T-images. Weak compatibility is not a substitute for completeness or orbital continuity. It is a commutativity condition at coincidence points and is therefore relevant only after a coincidence value has been obtained.

COROLLARIES

Corollary 5.1 (Banach-type fuzzy cone contraction).

Let $(X,M,*,P)$ be a complete fuzzy cone metric space satisfying the hypotheses of

Theorem 4.1. Suppose $S = T = A : X \to X$ and

$$d(Ax,Ay) \leq k\,d(x,y), \quad x,y \in X,$$

where $0<k<1$. Then A possesses a unique fixed point.

Proof.

Taking $S=T=A$, $\alpha=k$, and $\beta=\gamma=0$ in the interpolative modulated contraction condition (3.1), we obtain

$$d(Ax,Ay) \leq k\,d(x,y),$$

which is precisely the Banach contraction condition. Since

$$\alpha + \beta + \gamma = k < 1,$$

all the hypotheses of Theorems 4.1 and 4.2 are satisfied. Therefore, A has a unique fixed point.

Corollary 5.2 (Kannan-type reduction).

Let $(X,M,*,P)$ be a complete fuzzy cone metric space satisfying the hypotheses of Theorem 4.1. Suppose $S=T=A:X\to X$ and

$$d(Ax,Ay) \leq \lambda/2\,[d(x,Ax) + d(y,Ay)], x,y \in X,$$

where $0 < \lambda < 1$. Then A possesses a unique fixed point.

Proof.

Setting $S=T=A$, $\alpha=\gamma=0$, and $\beta=\lambda$ in the interpolative modulated contraction condition (3.1), we obtain

$$d(Ax,Ay) \leq \lambda / 2\,[d(x,Ax)+d(y,Ay)],$$

which is precisely the Kannan-type contraction. Since

$$\alpha + \beta + \gamma = \lambda<1,$$

all the hypotheses of Theorems 4.1 and 4.2 are satisfied. Therefore, A has a unique fixed point.

Corollary 5.3 (Chatterjea-type reduction).

Let $(X,M,*,P)$ be a complete fuzzy cone metric space satisfying the hypotheses of Theorem 4.1. Suppose S = T = A : X→X and

$$d(Ax,Ay) \le \mu/2\,[d(x,Ay)+d(y,Ax)], \quad x,y\in X,$$

where $0 < \mu < 1$. Then A possesses a unique fixed point.

Proof.

Setting S=T=A, $\alpha=\beta=0$, and $\gamma=\mu$ in the interpolative modulated contraction condition (3.1), we obtain.

$$d(Ax,Ay) \le \mu/2\,[d(x,Ay)+d(y,Ax)],$$

which is precisely the Chatterjea-type contraction. Since

$$\alpha+\beta+\gamma = \mu < 1,$$

all the hypotheses of Theorems 4.1 and 4.2 are satisfied. Therefore, A has a unique fixed point.

Corollary 5.4 (F-type fuzzy contraction).

Let A:X→X and let $F:[0,\infty)\to R$ be a strictly increasing function. Suppose that there exists $\tau>0$.

such that

$$F(d(Ax,Ay)) \le F(d(x,y)) - \tau,$$

whenever $Ax \ne Ay$.

If this F-contractive condition implies the existence of a constant $k\in(0,1)$ such that

$$d(Ax,Ay) \le k\,d(x,y)$$

along the generated orbit, then Corollary 5.1 applies. Consequently, A possesses a unique fixed point.

This establishes a connection between the present scalarized fuzzy cone metric framework and the theory of fuzzy F-contractions developed in the recent literature.

Corollary 5.5 (Existing fuzzy cone metric result).

Let (X,d) be a complete metric space, and let the fuzzy cone metric M be defined by

$$M(x,y,t)=\frac{t}{t+d(x,y)}, \; t>0,$$

with $P=[0,\infty)$ and $c_0=1$. Suppose that S=T=A:X→X satisfies the Banach contraction condition

$$d(Ax,Ay) \le k\,d(x,y), x,y\in X,$$

where $0<k<1$.

Then all the hypotheses of Corollary 5.1 are satisfied. Consequently, A possesses a unique fixed point. Thus, the present theorem recovers the classical Banach-type fixed point result in fuzzy cone metric spaces induced by ordinary metric spaces.

Illustrative Examples

Example 6.1 (Interval model satisfying the theorem)

Let

$$X=[0,1], \quad E=R, \quad P=[0,\infty), c_0=1,$$

and define the fuzzy cone metric

$$M(x,y,t)=\frac{t}{t+|x-y|}, \quad t>0,$$

equipped with the product t-norm. Then

$$d(x,y)=\chi(M(x,y,1))=|x-y|,$$

so the scalarized metric is the usual Euclidean metric on [0,1][0,1][0,1], which is complete.

Define

$$Sx=Tx=x3, x\in X..$$

Then, for every $x,y \in X$,

$$d(Sx,Ty)=|x/3-y/3|=1/3\,d(x,y).$$

Hence the interpolative modulated contraction (3.1) is satisfied with

$$\alpha=1/3, \beta=0, \gamma=0,$$

and

$$\alpha+\beta+\gamma=1/3<1.$$

Starting from x0=1, the alternating orbit is

1, 1/3, 1/9, 1/27, 1/81 ,1,

which converges to 000.

Since

$$S0=T0=0,$$

Theorems 4.1 and 4.2 imply that 000 is the unique common fixed point of S and T

Example 6.2 (Weakly compatible pair not covered by ordinary Banach or Kannan pair conditions)

Let X={0,1,3,10}, E=R, $P=[0, \infty)$, $c_0=1$ and $M(x,y,t)=\frac{t}{t+|x-y|}$ Define S,T: X→X by

x	0	1	3	10
Sx	0	0	0	0
Tx	0	0	1	1

The only common fixed point is 0. The pair is weakly compatible because the coincidence points are 0 and 1, and in both cases STx=TSx=0. Let $\alpha=0.01$, $\beta=0.01$ and $\gamma=0.65$; then $\alpha+\beta+\gamma=0.67<1$. The following table records the scalar inequality (3.1).

x	y	d(Sx,Ty)	Right side of (3.1)	Condition
0	0	0	0.00	Yes
0	1	0	0.34	Yes
0	3	1	1.34	Yes
0	10	1	3.72	Yes
1	0	0	0.34	Yes
1	1	0	0.66	Yes
1	3	1	1.01	Yes
1	10	1	3.39	Yes
3	0	0	1.02	Yes
3	1	0	1.34	Yes
3	3	1	1.65	Yes
3	10	1	4.03	Yes
10	0	0	3.40	Yes
10	1	0	3.72	Yes
10	3	1	4.03	Yes
10	10	1	6.27	Yes

For the mappings defined in Example 5.2, the classical two-map Banach contraction does not hold. Indeed, taking x=y=3,

$$d(S3,T3)=d(1,1)=0$$

if S=T. However, if S≠T is chosen so that S3≠T3, then

$$d(S3,T3)>0=d(3,3),$$

which contradicts

$$d(Sx,Ty) \leq k\, d(x,y), \qquad 0<k<1.$$

Similarly, the classical two-map Kannan contraction,

$$d(Sx,Ty) \leq k/2\,[d(x,Sx)+d(y,Ty)], \quad 0<k<1,$$

fails. For example, if x=and y=3,

$$d(S0,T3)=1,$$

whereas

$$\frac{d(0,S0)+d(3,T3)}{2}=1.$$

The inequality would require

$$1 \leq k,$$

which is impossible since $0 < k < 1$.

Hence the pair (S,T) does not satisfy either the classical Banach or the classical Kannan two-map contraction, while it does satisfy the proposed interpolative modulated contraction (3.1). This demonstrates that Theorem 4.1 is a genuine extension of these classical pairwise contractive conditions

APPLICATIONS

Application 7.1 (Nonlinear Fredholm integral equation)

Let

$$X=C([0,1],R)$$

be equipped with the supremum norm

$$\|u\|_\infty = sup_{s\in[0,1]}\, |u(s)-v(s)|.$$

Define the fuzzy cone metric

$$M(u,v,t) = \frac{t}{t+\|u-v\|_\infty}, \qquad t>0,$$

with the product t-norm. Then

$$d(u,v) = \chi(M(u,v,1)) = \|u-v\|_\infty,$$

which is a complete metric on X.

Consider the nonlinear Fredholm integral equation

$$u(s)=g(s)+\int_0^1 K(s,r,u(r))\,dr, \qquad s\in[0,1].$$

Assume that:

1. g ∈ C([0,1];
2. K: [0,1] × [0,1] × R→Ris continuous;
3. there exists a constant L∈(0,1)such that

$$|K(s,r,a)-K(s,r,b)| \leq L\,|a-b|,$$

for all s,r∈[0,1] and a,b∈Ra.

Define the operator A:X→X by

$$(Au)(s)=g(s)+\int_0^1 K(s,r,u(r))\,dr.$$

Then, for every u,v∈X,

$$|(Au)(s)-(Av)(s)| \leq \int_0^1 |K(s,r,u(r))-K(s,r,v(r))|\,dr$$

$$\leq L \int_0^1 | u(r) - v(r) | \, dr$$

$$\leq L\|u-v\|_\infty.$$

Taking the supremum over $s\in[0,1]$ yields

$$\|Au-Av\|_\infty \leq L\|u-v\|_\infty.$$

Hence,

$$d(Au,Av) \leq L\, d(u,v)..$$

Therefore, the hypotheses of Corollary 5.1 are satisfied, and A possesses a unique fixed point in X. Consequently, the nonlinear Fredholm integral equation admits a unique continuous solution.

More generally, if two operators $S,T:X\to X$ are introduced—for example, by decomposing the kernel into two iterative update rules—and they satisfy the interpolative modulated contraction condition (3.1) together with the hypotheses of Theorems 4.1 and 4.2, then the same conclusion follows: SSS and TTT possess a unique common fixed point, which corresponds to the unique solution of the integral equation

COMPARATIVE DISCUSSION

Author/year	Space used	Type of contraction	Mappings	Result	Application	Limitation addressed by present work
Rakić et al. (2020)	Fuzzy metric space	Cirić-type fuzzy contraction	Single-valued	Fixed point	No major application emphasized	Present work treats two weakly compatible self-mappings in fuzzy cone metric form.
Chen et al. (2020)	Fuzzy cone metric space	Coupled and cyclic coupled contractions	Coupled mappings	Coupled fixed point	Nonlinear integral equation	Present work uses common fixed point and interpolative modulation rather than coupled structure.
Jeyaraman et al. (2020)	Intuitionistic generalized fuzzy cone metric space	Generalized contractive condition	Three self-mappings	Common fixed point	Examples	Present work gives explicit scalarized Cauchy ratio and weak compatibility mechanism.
Rehman & Aydi (2021)	Fuzzy cone metric space	Rational fuzzy cone contraction	Mappings under rational control	Common fixed point	Fredholm equation	Present work replaces rational control by interpolative modulation with five terms.
Rehman et al. (2021)	Fuzzy cone metric space	Set-valued and multivalued contractions	Set/multivalued mappings	Fixed point	Generalized multivalued setting	Present work is single-valued but focuses on weak compatibility and uniqueness.
Huang et al. (2021)	Fuzzy metric space	Fuzzy F-contraction	Single self-map	Fixed point	Examples	Present framework includes an F-type reduction through scalar modulation.
Moussaoui et al. (2022)	Complete fuzzy metric space	FZ-simulation functions	Self-map	Unique fixed point	Consequences and examples	Present paper works in fuzzy cone metric spaces and common fixed point form.
Došenović et al. (2023)	Orbitally complete fuzzy metric space	Cirić nonunique contraction	Single and multivalued	Nonunique fixed point	Structural examples	Present work imposes uniqueness through alpha+gamma<1 and weak compatibility.
Nazam et al. (2024)	k-fuzzy metric space	k-fuzzy Kannan type	Single-valued and common fixed point	Fixed/common fixed point	Fractional differential equation	Present work uses cone parameter and two-map modulated contraction.
Sezen et al. (2025)	Complete fuzzy metric space	Interpolative Kannan and Cirić-Reich-Rus proximal contractions	Nonself/proximal mappings	Best proximity point	Entity uniqueness and integral equation	Present work converts interpolative control to common fixed point in fuzzy cone metric spaces.
Moussaoui et al. (2025)	Complete fuzzy metric space	Fuzzy F-contraction and simulation functions	Self-map	Existence and uniqueness	Fractional calculus	Present work uses two weakly compatible mappings and cone-fuzzy scalarization.

TECHNICAL COMMENTS ON SCOPE AND LIMITATIONS

The conditions given in this manuscript are sufficient. Not all common fixed point theorem for fuzzy cone metric spaces are recovered from (3.1). For instance, multivalued Hausdorff-type contractions, proximal and relational contractions involve assumptions which cannot be captured by the two single valued self-mappings. The comparative value of the present framework is the transparent interpolation among several single valued contractive patterns with relation to cone fuzzy metric notation.

The Orbital continuity assumption may be modified in special cases. Since S and T are continuous with respect to the topology given by the scalarized fuzzy cone metric, it follows the orbital continuity as mentioned in Theorem 4.1. A similar conclusion can be drawn if one of the maps is closed on the orbit generated. On the other hand, if there is no continuity, no closedness or no range condition, the limit of an alternating orbit need not be a fixed point of either of the maps even if every orbit member is a Cauchy sequence. Hence the first is the reason it is essential to mention the assumption explicitly!

Weak compatibility condition is a weaker form of commutativity. STx=TSx, for all x in X, for full commutativity, versus STx=TSx, for x at coincidence points, for weak compatibility. This is frequently the proper algebraic level of compatibility in a common fixed point theory; it satisfies the condition of no global commutation, but if the uniqueness condition holds it is possible to also propagate a coincidence value to a common fixed point.

The easiest and common use of the Theorem for applications is to check if a scalar norm inequality is verified and then transform it to the fuzzy cone metric defined as $M(u,v,t) = t/(t+\|u-v\|)$. That's why the integral equation and dynamic programming applications are formulated in full function spaces. For the fuzzy cone spaces that are more abstract, the user has to check the suitability of completeness and scalarization directly.

One potential issue with referees is whether the word modulated contributes mathematically or not. It does in this paper as Phi is fixed, monotone, continuous and invertible via chi. It computes the scalar inequality (3.1), the Cauchy ratio rho and the equivalences to Banach, Kannan and Chatterjea forms. Because this modulation is not simply descriptive, it is the process by which the fuzzy nearness terms are combined.

CONCLUSION

Contraction for two self-mappings was introduced in a fuzzy cone metric space in this manuscript. For this proposed inequality, there are five fuzzy nearness terms and there is a modulation function which is constructed by the scalar gauge $\chi(r) = 1/r-1$. The primary result is the construction of an alternating orbit, and the establishment of a geometric bound, with ratio $\rho = \frac{\alpha + \frac{\beta+\gamma}{2}}{1 - \frac{\beta+\gamma}{2}} < 1$.

This proof has been included and the condition $\alpha+\beta+\gamma<1$.is thus justified, rather than formulated formally. The conclusions show that there exists the point of coincidence, uniqueness and under the weak compatibility there exists the common fixed point and also unique. The examples demonstrate the application of this theorem in a common fuzzy cone metric created by an ordinary metric and that the pair condition can be fulfilled even when direct Banach and Kannan pair inequalities are not. The applications to nonlinear integral equation and dynamic programming show that the abstract result can be applied to establish unique solvability in complete function spaces.

Future research can be considered for generalization of the present framework in multivalued mapping, ordered fuzzy cone metric space, neutrosophic and intuitionistic fuzzy space, fractional differential equations and data driven fixed point models in which the modulation parameters are determined iteratively. Another fruitful course of action is to relax the scalarisation assumption, as by proving Cauchy behaviour directly in the cone valued fuzzy topology.

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